STUDENTS' UNDERSTANDING OF GEOMETRY: THEORETICAL PERSPECTIVES

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> Of all the decisions one must take in a curriculum development project with respect to choice of content, usually the most controversial and least defensible is the decision about Geometry.
>
> (CSMP staff, 1971, p. 281)

This comment, which questions the very place of Geometry in a modern syllabus is, itself, controversial. Today, with an emphasis on "value for money" education and the push for schooling to be relevant and have direct applicability to employment, the point being made is as current as it was in 1971. Nevertheless, Geometry continues to occupy a major part in the secondary and primary curriculum and this role, in NSW at least, has been expanded.

Research interest in Geometry has also grown considerably during the 80s. Further, there appears no sign of this trend abating in the 90s. One major catalyst for this sustained research effort can be traced to the ideas of the two dutch mathematics educators, Pierre van Hiele and Dina van Hiele-Geldof. Their work, which has come to be known as the van Hiele Theory, has helped shape and direct much of the research investigations associated with Geometry around the world.

This paper addresses students' understanding of Geometry by:

1. providing an overview of the van Hiele Theory;

2. considering research, from Australia and overseas, that reflects aspects of the Theory;

3. describing how the SOLO Taxonomy of Biggs and Collis can help clarify and extend ideas in the van Hiele Theory; and finally,

4. identifying future research themes.

Before starting on these central issues it is valuable to reflect on what 'theory' might mean to different researchers. This not only helps place in perspective why certain workers take particular paths but can also help explain why research can be interpreted in very different ways by various investigators.

One useful, but simplistic, view of research paradigms is to consider a single dimension with two poles. At one end there is the traditional scientific (or quantitative) paradigm and at the other end we have the phenomenological (or qualitative) paradigm. Researchers who ascribe to the quantitative end of the spectrum: look for context free generalisations, focus on individual variables and factors, and see a separation between facts and values. Social reality is seen to be objective and external to the individual. Most research designs are of a standardised nature and there is usually a strong reliance on statistical analysis to help quantify effects and causes. Obviously there are strengths and weaknesses with this philosophy of research. Work can be very precise, extraneous variables can be controlled through a variety of techniques, and hypotheses can be generated and tested. On the other hand the very notion of the need to quantify each aspect tends to highlight a mechanistic ethos which has the potential to denigrate human individuality. Also when quantification becomes an end in itself the results are often trivial and the sacrifices made to achieve measures can result in artificial situations which are unrelated to real life.

The qualitative paradigm is context based. The physical and social environment is seen to have a bearing on human behaviour. The emphasis is on the importance of subjective experience. As a result there is a more holistic approach to data collection. The research designs are usually flexible and there is a reliance on narratives or interviews. The strengths of such an approach are the identification of unexpected outcomes and relationships. Many subtleties and complexities hidden in the quantitative approach can be revealed. These strengths, however, are often at the expense of suitable measures of validity and reliability. Consequently the opportunity to generalise findings is reduced.

As mentioned earlier, this dichotomy represents a simplistic view of research as it is possible for a researcher to select elements from anywhere on the continuum to help answer specific questions. Nevertheless, it does help to explain some underlying philosophies of research and researchers.

My case highlights the deficiencies in the simple, one-dimensional continuum. I see myself down the scientific paradigm end and yet most of my recent research has involved working with students in schools on the topics they are being taught. There are many phenomenological elements to this work. However, central to my thinking is that there can be theories which explain student growth and that the testing and development of such theories should be a major aspect of any research. I would go so far as to say that testing of research questions derived from theories is of critical importance. Investigations undertaken in the absence of theory are almost certain to leave one as uniformed at the end as one was at the beginning.

For a theory to be useful it should be able to be tested and offer new questions that, in the absence of the theory, might not have been asked. It should also be able to satisfy at least three criteria:

1. Explanation, that is, offer reasons for what is observed;

2. Unification, that is, synthesise or link together previous work;

3. Prediction, that is, offer insights into new areas that have not been explored.

For me, in the early 80s, the van Hiele Theory satisfied these conditions and offered me the type of framework I was seeking. Now, while I feel I have moved on (or grown further) in this area I am cognisant of the legacy of the van Hiele Theory in developing my ideas.

THE VAN HIELE THEORY

The van Hiele Theory was developed in the 1950s when Pierre and Dina undertook companion PhDs (that is, they worked in a similar area of research, namely, Geometry but on different aspects) at the University of Utrecht. Their work evolved out of their experiences as teachers where they found secondary school Geometry difficult to teach.

Dina's work involved a teaching experiment aimed at developing students' understanding of Geometry. Pierre's thesis developed the notion of levels of thinking and the features that would lead to students exhibiting insight in Geometry.

Dina died in 1959 and Pierre has continued to develop the theory. His latest book *Structure and Insight* was published in 1986 and brings together the main themes which make up the Theory.

Van Hiele's ideas have much in common with those of Piaget in that they ascribe student understanding to a series of levels or stages. However, there are important differences between the two theories. For example, the van Hiele Theory:

i) places great importance on the role of language in moving through the levels;

ii) concentrates on learning rather than development, hence the focus is on how to help develop student understanding; and,

iii) postulates that ideas at a higher level result from the study of the structure at the lower level.

These features combine to help the theory stand apart.

In overview the theory has two main aspects which combine to provide a philosophy of mathematics education (not only of Geometry). The two key aspects are:

1. A series of levels that students grow through in acquiring competence and understanding;

2. Five teaching phases that assist students to move from one level to the next.

By taking into account both of these aspects, students can exhibit and develop what the van Hieles refer to as 'insight'. A student would indicate insight when he/she could act **adequately** with **intention** in a **new situation**. For this to happen it is clear that students must have ownership of their mathematics and be capable of addressing questions they have not been given previously .

This section considers the van Hiele Theory. In particular, the following questions are addressed:

1. What are the van Hiele levels of growth?

2. What features or properties underpin these levels?

What are the van Hiele levels of growth?

Van Hiele identified five levels of development. I will use the numbers 1 to 5 to identify the different levels and the word descriptions of Hoffer (1981).

Level 1 (Recognition)

Figures are judged by their appearance. A figure is recognised by its form or shape. The properties of a figure play no explicit role in the identification of a figure.

Level 2 (Analysis)
Figures are identified by their properties. The properties, however, are seen to be independent of one another. For example, the properties are not organised in such a way that students will realise that a square is a rectangle.

Level 3 (Ordering)
The properties of figures are no longer see as independent. There is seen to be an ordering of the properties with one property preceding or following another property. Relationships between different figures are also understood.

Level 4 (Deduction)
The place of deduction is understood. Necessary and sufficient conditions can be employed. Proofs can be carried out and not simply learned by rote. Definitions can be devised.

Level 5 (Rigour)
Comparison of various deductive systems can be undertaken. Different geometries can be explored based upon various systems of postulates.

Clearly, the levels represent a logical hierarchical arrangement with the substance of each level flowing from an analysis or investigation of the previous level. The recognition of a figure, a Level 1 feature, is an essential prerequisite for Level 2. The consideration of properties at Level 2 will eventually lead to Level 3 understanding where students see relationships between them, i.e., how one or two properties may lead to a third. In much the same way different figures can be seen to share common sets of properties. It is at this level that a square can be considered as a rectangle, rhombus, or parallelogram.

For Level 4 it is the relationships that become the focus of study. This leads to the development of theorems. In addition, there is now the opportunity to consider which relationships or properties are necessary and sufficient to describe a figure. From this, students can now generate definitions. Of course, Level 5 understanding evolves out of an analysis of the theorems and postulates at Level 4.

IMPORTANT FEATURES OR PROPERTIES OF THE LEVELS

Several properties are associated with the levels of thinking.

1. Students may be on different levels for different concepts. However, once one concept has been raised to a higher level, it will take less time for other concepts to reach that level.

2. To move a pupil from one level to the next, except maybe for exceptional students, requires direct instruction, exploration and reflection by the student. However, it takes time to move from one level to the next; teachers must be prepared to allow time for this growth to occur. Dina wrote of 50 lessons needed to move 12 year olds from Level 2 to Level 3.

3. A student cannot attain a higher level without first passing through the lower level(s). However, pupils can simulate higher levels by learning rules or definitions by rote or by applying routine algorithms that they do not understand.

4. Students need to confront a personal "crisis of thinking" in moving from one level to the next. They cannot be forced to think at a higher level. However certain teaching strategies can inhibit such growth and place boundaries on students' potential.

5. "Level reduction" occurs when structures at a higher level are re-interpreted at a lower level. This usually occurs by making the structures at the higher level visible, for example, the use of Z or F to help students 'understand' alternate and corresponding angles. The effect of this procedure, when it is teacher directed, can be counter productive as it can remove the stimulus for the students to attain a higher level.

6. Each level has its own language or linguistic symbols. People reasoning on different levels speak different languages and in general cannot understand one another. This language problem can occur even between students within the same classroom. Thus very serious communication problems exist between students on one level and their fellow students , teachers, text-books, and exercises on another level.

7. Each level has its own organisation of relationships. Teachers need to be aware that what may appear to be correct at one level may not be seen to be correct at a higher level. The most obvious example of this is that it is not until Level 3 that a square is seen to belong to the set of rectangles.

8. Concepts implicitly understood at one level become explicitly understood at the next.

9. The learning process is discontinuous. That is, a student having reached a given level remains at the level for a time as if maturing. Forcing a student to perform at a higher level or directing your teaching at a higher level will not succeed until the maturation process has occurred.

10. Rote learning or applying routine algorithms without understanding is no level.

RESEARCH ON THE VAN HIELE THEORY

Research on Levels

Most research on the van Hiele Theory has been directed at the notion of levels of understanding. While there has been enormous empirical support confirming the levels there have been problems. Usiskin (1982) used a multiple choice test across a number of topic areas in Geometry and found that 75% of students could be allocated to a level. This test has been used by over 100 other investigators (Usiskin and Senk, 1990). One typical example was by Chaiyasang (1987) who used the test on over 3000 students from Year 6 to 9 in eastern Thailand. He found that 90.1% of the students tested could be allocated a level.

However, I believe the test has a basic flaw. It assumes that students will be at the same level for different concepts. This is contrary to the van Hiele Theory and in direct opposition to Mayberry's (1981) research in which she found students were of different levels in different concepts. (This point has also been recently verified with Australian students by Lawrie (in press).) Other researchers (e.g., Crowely, 1990) have also questioned the test for its validity and reliability. The reason that such strong results are being identified in spite of the problems mentioned is an interesting research question in itself.

Other problems identified with the level descriptions are (i) the difficulty in testing Level 5, and (ii) the need for some level (Level 0) below the base level identified by van Hiele. In the former case Usiskin (1982) found little help in the writings of van Hiele in helping list behaviours that were suitable for Level 5. He concluded his study with the observation that "Level 5 either does not exist or is not testable" (p. 79).

The need for a level (or levels) below van Hiele's base level has been identified by several writers (Mayberry, 1981, 1983; Usiskin, 1982; Senk, 1989; Pegg and Davey, 1989). These studies have identified numbers of students who cannot satisfactorily meet van Hiele's Level 1 criteria. A possible reason for this anomaly could be that van Hilele's work is concerned primarily with secondary students. This being the case, then his base line would be appropriate in that instance. As researchers have extended his ideas and worked with younger children the inadequacies of his lowest level have become more apparent.

Finally, Pegg and Davey (1989) have suggested that it is valuable to consider van Hiele's Level 2 as two separate levels. The first of these is characterised by a student focusing on only one property. The second involves the identification of two or more properties. The reason for the split arises out of the large numbers of students, in Years 5 - 7, who are able to spontaneously provide only one property in descriptions - despite extensive probing on the part of the interviewer. This finding is consistent with predictions of the SOLO Taxonomy and is discussed later in the paper.

Research on Properties

Several of the properties or features listed previously, about the van Hiele Theory, have been addressed in research studies. The most significant overseas papers and reports of the 80s and 90s are by Usiskin (1982); Fuys, Geddes and Tischler (1985); Mayberry (1983); Burger and Shaughnessy (1986); Bobango (1987); Denis (1987); Senk (1989); Gutiérrez, Jaime and Fortuny (1991). The numbering below follows that established earlier:

1. **Different levels for different concepts**
   The work of both Mayberry (1981, 1983) and Burger and Shaughnessy (1986) has identified students to be on different levels for different concepts. Pegg and Davey (1989) found that, even on activities based on closely related figures such as a rhombus and a square, there were often differences. These differences can be explained in terms of student's familiarity with the shapes. However, the differences identified were not great. Hence it seems that while a student's level of understanding of various concepts might not be the same they are clearly not independent.

2. **Growth takes time**
   Dina van Hiele-Geldof (1984) indicated that it took her some 20 and 50 lessons to move a class to Level 2 and Level 3, respectively. While there has been no direct replication of her work, there is a strong indication that considerable time is required for student growth. Fuys *et al* (1985) found six to eight 45 minute sessions was

not sufficient time for the students in their study to grow to the next level. In fact a number of students showed little or no progress from their starting level.

3. Hierarchy

The research evidence on whether levels form a hierarchy has been uniformly positive. Mayberry (1981, 1983) using Guttman's scalogram analysis found that the levels do form a hierarchy (reproducibility = .97). These results were repeated in a study (Denis, 1987) using Mayberry items on a sample of Puerto Rican adolescents (reproducability = .99). De Villiers (1987) reporting on a study in South Africa cited a reproducibility coefficient of .90. The results of other writers, e.g., Fuys *et al* (1985), where some students under instruction at Level 1 are seen to be moving towards Level 2 and students at Level 2 are seen to be moving towards Level 3, can also be interpreted as verifying the hierarchical nature of the levels.

4. & 5. Crisis of thinking and level reduction

I could find no specific research related to either of these two areas. Both these areas are critical to teachers and curriculum planners. If a crisis of thinking is necessary for a student to move from one level to the next then the characteristics of such behaviour should be explored. Further, if level reduction techniques do result in long-term disadvantages for students and prevent them from facing a crisis of thinking then examples of such behaviour need to be identified.

6. Language and levels

As part of their study, Fuys *et al* (1985) considered several mathematical texts available in the US market. They found that: (i) the texts were primarily concerned with Level 1 and Level 4 activities; (ii) often three levels were addressed at once; and, (iii) if teachers do not make language an important aspect of teaching then they can set up a barrier at Level 2 for student growth. Other studies have also commented on the language used by students at various levels (e.g., Burger and Shaughnessy, 1986). Pegg and Davey (1989) noticed that certain words and word combinations had strong links with various levels. For example, students at Level 1 and low Level 2 used words such as "even" and "corner" instead of "equal" and "angle". Phrases such as "opposite sides equal" were found to be only associated with high Level 2 or Level 3 understanding.

7. Different perceptions at different levels

Most studies have found data to support the notion that students on different levels think differently. However, the findings associated with Level 3 thinking in some studies need to be challenged. It is not sufficient to say that a student is not at Level 3 if he/she does not believe a square is a rectangle. Class inclusion is not simply a part of a natural mathematical development. It is linked very closely to a teaching/learning process. It depends upon what has been established as properties. An interesting comparison is between a trapezium and a parallelogram. In this case, the acceptance of class inclusion depends on whether **only one pair** of equal sides is a property of a trapezium. If it is, then a parallelogram does not belong to the set of trapeziums. The main feature of Level 3 should not, in my view, be the acceptance of class inclusion but the willingness, ability and the perceived need to discuss the issue.

8. Implicit - Explicit
A major feature of the van Hiele Theory is that what is implied at one level is made explicit at the next. The hierarchy research has a bearing here. So too has the work of Vygotsky (1976). In particular, his notion of the zone of proximal development (ZPD) is relevant. In a practical situation this can be addressed in interview by the techniques of probing and prompting. A student's level is basically what they can offer spontaneously to some stimulus or question. To help clarify the issue, probing questions, i.e., ones that let the student know that you require more information, are valuable. They can take the form of "Can you tell me any more?" "Would you add further to your comments?" "Can you clarify your comments?". Prompting questions, on the other hand, enable the interviewer to provide the student with additional cues. After a time these cues cease to stimulate useful information. The differences between these two forms of questions is the area of thought which Vygotsky refers to as ZPD. I see it as a useful way of finding what implicit knowledge a student processes and what knowledge the student can be expected to acquire in the near distant future.

9. Discreteness of levels
This area of research has been commented on by several writers but has not been the focus of many investigations. There appears to be growing acceptance that the levels are more dynamic in nature than static. The research of the Gutiérrez team has extended this idea by conceiving the notion that students might be growing in several levels at the one time. This does not negate the hierarchy proposed by van Hiele. They see that some "critical mass" acquisition of a lower level is needed before growth into a higher level is obtained. Despite some problems with their design (see Lawrie (in press)) their work does open up the potential for a more detailed analysis of what it means to be 'at' a level. My own work identifies several features:

i) There are transition periods between the levels. This tends to show up growth as more of a continuum.

ii) Students entering a new level are most vulnerable and are prone to oscillate between the new and the old for some time.

iii) Within a given concept, and dependent on the teaching, students are relatively stable in their level of understanding at a given time.

iv) Students have a potential level. New concepts move relatively quickly to this level providing a plateau-type effect.

10. Rote learning - no level
While rote learning has been addressed extensively in the Psychology literature, I have identified no specific research directed at van Hiele's comment about rote learning. The role of rote learning in assisting or hindering new knowledge acquisitions is an important research question. Is rote learning at all valuable to understanding?

From the above, it is clear that the van Hiele Theory has been subjected to some research scrutiny in the past ten years. However, the investigations have addressed only limited aspects of the theory. Also, most research has been directed at global issues. Usiskin's

(1982) comment that "many mathematics educators are accepting and using this theory on the basis of characteristics of the theory rather than a testing of the individual components" (p. 7) is still relevant. The potential for more directed research which focuses on more specific features of the Theory is enormous.

THE SOLO TAXONOMY

Another framework that has evolved as a reaction to perceived short comings in Piaget's Theory is the SOLO Taxonomy of Biggs and Collis. Despite this common starting point, there are some differences between the formulations of the van Hiele Theory and the Solo Taxonomy. Primarily, van Hiele is about describing thinking processes while Biggs and Collis are interested in categorising the quality of an individual's response. In the latter case the analysis of the response made may lead to some understanding of the thinking processes, but it may not. Issues such as motivation, the form of the question, and prior knowledge in a specific area could all influence a person's answer. This could cloud or under-value the thinking processes used. Nevertheless, the two theoretical stances do share a number of common traits and useful information can be gleaned by using both perspectives when considering research findings. Before exploring this aspect it is valuable to review briefly, key aspects of the SOLO Taxonomy.

The SOLO Taxonomy uses two criteria when determining the level of a student's response to some question or problem. The first, referred to as the mode of functioning, is determined by the level of abstraction of the elements used. The second is the complexity of the structure within a given mode. Figure 1 summarises this information.

There are five modes and these can be compared to the stages outlined by Piaget. The modes are:

Sensori Motor (from birth). Here a person reacts to the physical environment. For the very young child it is this mode in which complex motor skills are acquired. In later life these skills evolve to be associated with various sports. This form of knowledge is described by Biggs and Collis as tacit knowledge, i.e., knowing how to carry out an action without necessarily being able to describe the act in detail.

Ikonic (from about 18 months). Here language emerges. A young child develops words and images which can stand for objects and events. In later life this form of functioning assists in the appreciation of art and music. Knowledge of this type is referred to as intuitive.

Concrete Symbolic (from about 6 years). Here symbol systems develop. A person expresses his/her thinking through the use of symbol systems such as written language and number systems. This mode represents a significant shift in the level of abstraction despite the links that tie the systems to the empirical world. This form of knowledge is referred to as declarative.

Formal (from about 16 years). Here abstract concepts such as principles or theories are considered. Students are no longer restricted to a concrete referent. As a result, the consideration of a range of possibilities and constraints is undertaken leading to the development of general cases.

Post Formal (from about 20 years). Here abstract concepts identified in the formal mode are challenged and questioned. Students are expected to advance knowledge in a discipline and would be seen to be associated with original research.

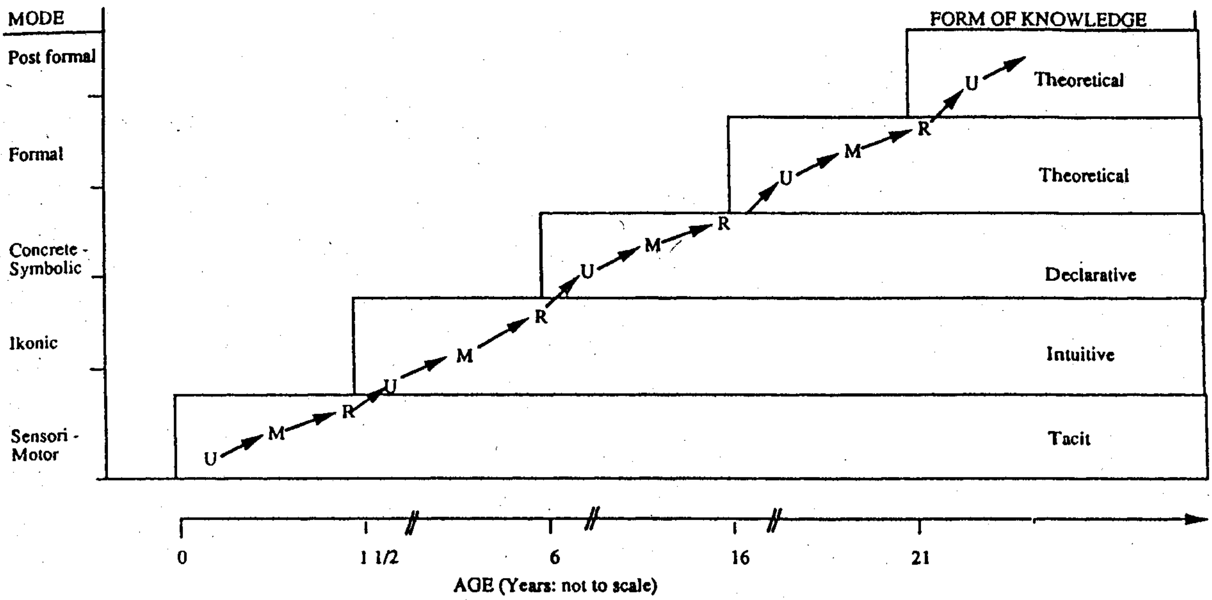

FIGURE 1: MODES, LEARNING CYCLES, AND FORMS OF KNOWLEDGE OF THE SOLO TAXONOMY

(adapted from Biggs and Collis Developmental Learning and the Quality of Intelligent Behaviour in H. Rowe (Ed) Intelligence: Reconceptualization and Measurement. Melbourne: ACER

The ages provided with each mode are those when the mode typically becomes available. Associated with each mode are a series of levels which provide a measure of the complexity of structure. These levels comprise basically three dimensions, namely:

1. How the response relates to cues or information in the question;

2. The amount of working memory or attention span required;

3. The need to (a) come to a conclusion (answer) referred to as closure and (b) be consistent and cope with contradiction (Biggs and Collis, 1982 pp.26-28).

There are five levels for each mode:

Prestructural. This level is below the mode of functioning required of the task. Examples would include a student refusing to answer a question, or providing a tautology. The amount of working memory used is small and there is a strong-felt need to reach a conclusion quickly and to ignore inconsistencies.

Unistructural. This is the first level within the "target" mode. It can be recognised by a student focusing on only one relevant aspect. The amount of working space used enables the student to select one cue and apply a relevant operation. The amount of working memory used is increased over the previous level but, because of selecting only one aspect, inconsistencies are common.

Multistructural. This is the second level within the "target" mode. It can be recognised by a student focusing on a number of relevant aspects but connections or relationships between the aspects are not made. The student closes when several aspects are used but inconsistencies can occur because of the perceived independence of the aspects.

Relational. This is the third and last level within the "target" mode. It is recognised by a student being able to generalise within a given or experienced context using related aspects. The working memory used needs to consider the elements and their inter-relationships. Closure is delayed until this is accomplished. The result is usually consistent within the specified context but may not be consistent across different contexts.

Extended Abstract. This level lies outside of the "target" mode and can be equated to the unistructural level of the next mode. It is recognised by students being able to generalise to situations not experienced. Inconsistencies are able to be resolved or tied to specific circumstances. One result of this is that answers to questions may seem to be left open. Clearly the working memory required for such a response is large.

THE VAN HIELE THEORY AND THE SOLO TAXONOMY

There have been three attempts in the research literature to consider the van Hiele Theory and the SOLO Taxonomy together. Olive (1991) chose to integrate the two theories by considering them in relation to Skemp's (1979) model of mathematical understanding. While his work was related to Geometry, the basis of his research was directed at ninth grade students' Logo programming skills.

The other two studies were by Jurdak (1989) and Pegg and Davey (1989). Pegg and Davey (1989) described a study in which the two theories were compared. The particular focus of the comparison was directed at Levels 1 to 3 in the van Hiele Theory. The findings of the study were at variance with those of Jurdak (1989) who also compared the two theories. The remainder of this paper seeks to explore the differences in the two papers. First, a comparison of the findings found in Pegg and Davey and Jurdak are discussed and second, reports on two further studies are included in which the results add further weight to support the original findings of Pegg and Davey.

Jurdak (1989) took an example from Burger and Shaughnessy (1986) which involved identifying quadrilaterals from a set of figures consisting of squares, parallelograms, and rhombuses. The students' answers were classified in terms of van Hiele levels. Jurdak took this information and reclassified it according to the SOLO Taxonomy. In summary, the results for the lower three van Hiele levels showed that: Level 1 (van Hiele) was equivalent to the unistructural level in the SOLO Taxonomy; Level 2 (van Hiele) was equivalent to the multistructural level; Level 3 (van Hiele) was equivalent to the relational level. The one-to-one correspondence identified means that for Level 1 to be equivalent to the unistructural level, the single aspect of focus was the shape of the figure "as a whole but not its properties" (p. 158).

Pegg and Davey (1989) approached the task of comparison differently. They tested and interviewed approximately two hundred and seventy students from Year 3 to Year 7 about their understanding of four basic 2-D shapes. When the students' responses were analysed they could be divided into three clear groupings. Many students could describe the figures using only global features. Such responses were:

a rectangle is a long square,

a parallelogram is like a rectangle that has been pushed to one side.

A second group of responses listed properties of the various figures. Examples included:

a square has all four sides equal and the angles are all 90°,

a parallelogram has four sides two of them are equal to each other the other two are equal to each other. If you run the lines forever it will never meet.

However, most significantly, a large number of responses which made up a third group were identified that only contained one property and this did not change under careful probing. Examples of this included:

a square has four even sides,

a rectangle: two sides are the same and the top and bottom are the same,

a rhombus: it looks just like a diamond it has equal sides.

What became clear from the analysis was the large difference in quality between the two types of answers in these last two groups. Thus to consider the responses within only one mode (namely, the concrete symbolic), as Jurdak had done, was not doing full justice to the data. Instead, it seems accurate to categorise the descriptions which focused on global, visual criteria as belonging to the ikonic mode. When this is done, the descriptions which use only one property can be categorised as unistructural in the concrete symbolic mode. Of course those students whose responses were able to provide more than one property were classified as multistructural. It appeared from the interviews that, in general, the best that could be achieved by relatively young children was a multistructural response to the question that required descriptions to be given. The possibility that students whose responses were coded as multistructural might also be able to respond at the relational level (i.e. van Hiele's Level 3) could only be tested by asking additional questions.

There were also numerous examples of students who, having given one or a number of properties, felt the need to go further and provide some visually-based statement as if to support their answer. Examples of this included:

a parallelogram is just like a rectangle tilted over. The lines are parallel and the angles are different to a rectangle. It is similar to a rhombus,

a rhombus: has four sides of which are in pairs. Each side of a pair is facing the other so they are parallel. It looks like a square pushed over a bit to make a pair of lines slope. If you draw a diamond you have the right slope.

This tendency for students to use imagery to support their answer was a common feature. However, the van Hiele Theory makes no mention of the use of lower levels to support high levels. Van Hiele, implies that once a new level is attained it completely subsumes the previous level and that people can no longer use the lower level. The SOLO Taxonomy does not adopt this stance. Not only does it suggest that students can function within different modes on a particular task but the earlier modes continue to develop throughout a person's life. In the case of the examples above, the SOLO Taxonomy would categorise these responses as concrete symbolic with ikonic support.

In summary, the results showed that; (i) it is valuable to consider young students' descriptions of 2-D shapes within the framework offered by two modes of functioning, namely, ikonic and concrete-symbolic; (ii) Level 1 (van Hiele) represented a possible series of levels in the ikonic mode with the better answers which focused successfully on some global shape as being able to be classified at the relational level; (iii) there was no direct equivalent for the unistructural level (concrete symbolic mode) of the SOLO Taxonomy in the van Hiele theory. It is possible to surmise that van Hiele may have seen it as part of his Level 2; (iv) there appeared to be a direct equivalence between Level 2 and 3 in van Hiele's theory and the multistructural and relational levels in the concrete symbolic mode of the SOLO Taxonomy.

Two other studies undertaken by Davey and Pegg (1990, 1991) have a bearing on the results above. Both studies involved an intensive interview and testing programme. The first was directed at parallel concepts and the second at angle concepts. While each study explored many features, it is the descriptions provided by the students that will be the focus of the following discussion.

Students' descriptions of parallelism mirrored those identified above for 2-D figures. That is, three basic groupings of responses were identified. A number of young students gave simple descriptions based on some visual prototype. Examples included:

> looks like a sidewalk and a road,
>
> looks like a door.

These responses can be interpreted as Level 1 in van Hiele's terms or within the ikonic mode of the SOLO Taxonomy.

A second group of students focused on lines in a fairly general way. There appeared a gradual growth in their answers up to the point where the students were able to qualify their statement about straight lines by adding that "the lines do not meet". Examples of this growth included:

> lines on their side,
>
> 2 lines side by side, across, diagonal or straight,
>
> 2 straight lines that never meet and go on forever.

These responses are consistent with the classification of unistructural (concrete symbolic mode) in the SOLO Taxonomy as they focused on one property although there was a clear visual element to the answers.

The final group of descriptions was similar to the previous group except the answers were extended by including the additional fact that "the lines were equidistant". Examples from this group of responses included:

parallel means 2 straight lines apart that are the same distance between all the way,

two lines that do not meet and have the same distance in between.

This was seen to be the multistructural level (concrete symbolic mode) in the SOLO Taxonomy and Level 2 in the van Hiele Theory. To reach this answer a student had to first take into account the fact that the lines did not meet before they could focus on the equidistant property.

When students were asked about angles, four basic categories of responses were received. The first group of responses frequently used the word 'corner' and seemed to have no common understanding of corner or angle. Examples included:

a pointy triangle ... corners are sharp,

a corner is a point - an angle is the same like that in a cube.

The second group of responses brought in the idea of lines but in a very vague way. Examples included:

two things meeting together and it has to have a corner,

where a straight line ends and ends at the end of another line.

The third group focused on "where lines meet" and it was common for them to talk about special degree angles like 90°, 180° or 360°. Examples included:

a sharp bend in a line. The most common angle is 90° like the corner of a square,

Two straight lines meet together at a corner.

The fourth and final group of responses talked about the area or distance between two lines. Examples included:

distance between two lines,

length in degrees from one line to another.

The first group, with its strong visual focus and lack of specific detail, reflects levels in the ikonic mode or Level 1. The remaining three groups are, in order, the unistructural, multistructural and relational levels of the concrete symbolic mode or Level 2 and Level 3 of van Hiele's theory. It is important to realise that each of the groups of responses was associated in broad ways with various skills in angle problems. For example, it was not until group four above that students could consistently halve the size of a given angle or compare the size of different angles. These findings perhaps provide some insight into why students find angle concepts so difficult in that they need to be at the relational level (concrete symbolic mode) or Level 3 to be routinely successful.

Two other features deserve comment. First, the data of the studies described above also point to a further elaboration of the SOLO Taxonomy (see Pegg, 1992). Within the responses that focused on a single property, three distinct groups were identified. The first of these simply provided in their description some comment on lines or sides, e.g., "a square has four lines" or "a rectangle has four straight sides".

The second group attempted to qualify this description, as if they knew that just talking about lines/sides was insufficient. One example "a rectangle is similar to a square but the top and bottom are longer than the other two sides". The third group also focused on sides/lines but they were able to include the key characteristic, namely, equality, e.g, "a square has four even sides".

These groupings lead one to see an early cycle of growth, consistent with the descriptions of unistructural, multistructural and relational levels. The focus of this growth is towards identifying accurately a single property. This finding implies that there are two cycles (and possibly more) within the concrete symbolic mode.

Figure 2 below illustrates this diagrammatically. Here U_1, M_1, R_1, represents the first cycle and U_2, M_2, R_2 the second. The key feature being that R_1 and U_2 are equivalent. This means, the group of students' responses, coded initially as R_1, i.e., because they bring together all the aspects that make up the first property, can also be coded as U_2, i.e., a focus on one property.

Concrete Symbolic Mode

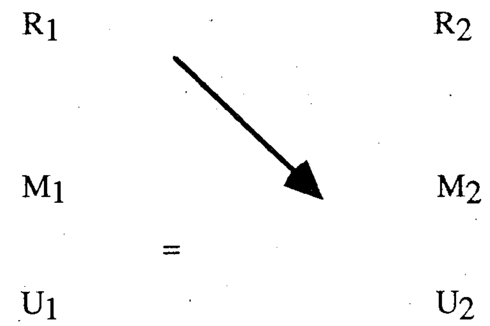

Figure 2: Relationships between cycles and levels within a mode.

This finding, about the possibilities of different cycles within a mode, offers new possibilities in understanding student growth in concepts.

Second, van Hiele (1986) has flagged a different interpretation for his levels that he believes might be of value to some people. In this he refers to: Level 1 - the First level as the **visual level**; Level 2 and Level 3 - the Second level as the **descriptive level**; and, Level 4 - the Third level as the **theoretical level**. However, he did not choose to explore this structure as it did not offer him the potential of his initial classification.

This suggestion is interesting for two reasons. It indicates that level descriptions are often context bound and that, depending on the purposes, the same data could well be interpreted in different ways. More importantly, however, is the significance of this different interpretation when a comparison is made with the SOLO Taxonomy. It would seem that what van Hiele has identified unknowingly by his three new levels are the ikonic, concrete symbolic, and formal modes of the SOLO Taxonomy.

FUTURE RESEARCH INITIATIVES

I have already commented, throughout the paper, on potential research questions. However, in addition to these, there do appear to be four main areas in which research is needed.

1. The ability range within Level 3 (van Hiele) or the relational level (concrete symbolic mode) seems to be very large in Geometry. Van Hiele talks of at least three aspects relevant to this level, namely: an ordering of properties; class inclusion of figures; and, simple deductions. Investigations are needed which explore relationships between these aspects and to what depth such understanding can be expected.

2. The work on the van Hiele Level 1 has been limited. By considering this level as a mode of functioning, i.e., the ikonic mode in SOLO terms, it opens up a field of investigation into young children's development of geometric concepts.

3. Similar to 2 above, but the focus is on Level 4 understanding. Again if this level is compared with the formal mode in SOLO it opens up the possibility of a more careful analysis of the way in which more capable students address advanced questions.

4. The notion of cycles of growth within the concrete symbolic mode has important implications to teaching. More information about these cycles, and the possible identification of further cycles, opens up the possibility that the SOLO Taxonomy may be an even more useful curriculum development tool.

CONCLUSION

In this paper I have attempted to explain, review and update an important research theme in the teaching and learning of Geometry. Clearly, it is not the only focus of research in Geometry. A quick look at the 'Research in Geometry and Measurement' Chapter in the recently published *Mathematics Education Research in Australasia: 1989-1991* will confirm this observation. Nevertheless, this area continues to occupy a large amount of research effort world wide. I find it a particularly exciting and challenging field of investigation. It is clearly a major frontier of research in mathematics education. One that has enormous potential to help us understand how students acquire concepts and one that can help us improve how and what we teach.

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